

# A stochastic process approach to multilayer neutron detectors

Dragi Anevski<sup>1</sup>, Richard Hall-Wilton<sup>2</sup>, Kalliopi Kanaki<sup>2</sup>, and  
Vladimir Pastukhov \* <sup>1</sup>

<sup>1</sup>Centre for Mathematical Sciences, Lund University, Lund  
SE-221 00, Sweden

<sup>2</sup>European Spallation Source ERIC, P.O Box 176, Lund  
SE-221 00, Sweden

## Abstract

The sparsity of the isotope Helium-3, ongoing since 2009, has initiated a new generation of neutron detectors. One particularly promising development line for detectors is multilayer gaseous detectors where the neutron conversion into charged ions in a thin <sup>10</sup>Boron-containing layer. In this paper, a stochastic process approach is used to determine the neutron energy from the additional data afforded by the multilayer nature of these novel detectors.

The data from a multi-layer detector consists of counts of the number of absorbed neutrons along the sequence of the detector layers, in which the neutron absorption probability is unknown. We study the Maximum Likelihood estimator for the intensity and absorption probability, derive conditions for their existence and show that when they exist, they are consistent and asymptotically Normal, as the experiment time (or number of incoming neutrons) goes to infinity. We combine these result with known results on the relation between the absorption probability and the wavelength, to derive an estimator of the wavelength and to show consistency and asymptotic normality for the estimator. We illustrate the methods on simulated and real data.

*Key words:* Maximum Likelihood, Poisson Processe, Neutron Scattering.

---

\*pastuhov@maths.lth.se

# 1 Introduction

Assume that a beam of neutrons arrives at the face of the detector. The detector consists of a sequence of 10Boron-layers, between which there are gas filled cavities. The principle of the detector can be described in a simplified manner as follows: A neutron that goes through a 10Boron layer can sometimes interact with a 10Boron atom in the layer, exciting the atom into an unstable state temporarily from which it will fall back to the stable state and thereby emit an electrically charged particle, that will ionize the gas. This electrical potential in the gas filled chamber is then detected, when it reaches a certain lower bound, the instrument will detect the charge and thereby note that a neutron has been absorbed. The outcome of this is that we have a count of +1 of the number of neutrons that have passed and been detected. The probability with which a neutron is absorbed and then detected is a function of the energy content of the neutron, i.e. a function of the neutron's wavelength.

If we see the neutron beam as a set of particles that hit the face of the detector each neutron will either be absorbed or not, at the first layer. If it is not absorbed at the first layer, it may possibly be absorbed at the second layer, and so on. It is sensible to model the incident neutron beam as a Poisson process, with unknown intensity. From the above simplified description of the detector it is clear that the data from a multi-layer detector will consist of counts of a number of absorbed neutrons along the sequence of the detector layers.

Each neutron is absorbed with a probability, that depends on the neutron's wavelength, and that therefore is unknown. It is also assumed that the Poisson process intensity is unknown. Given the data we can use a likelihood approach, and study the Maximum Likelihood estimator for the intensity and absorption probability. In the sequel we derive conditions for their existence and show that when they exist, they are consistent and asymptotically Normal, as the experiment time (or number of incoming neutrons) goes to infinity. We combine these result with known results on the relation between the absorption probability and the wavelength, to derive an estimator of the wavelength and to show consistency and asymptotic normality for the estimator. We illustrate the methods on simulated and real data.

By a beam we mean a stream of particles with a certain fixed wavelength  $\mu_0$ . Let the number of neutrons that arrives in the time interval  $[0, t]$  be denoted by  $X_0(t)$ . Then  $X_0(t)$  is a non-decreasing counting process, such that  $X_0(0) = 0$ .

A sensible model for the process of incoming neutrons  $X_0(t)$  is that of a Poisson process. This means that the times between consecutive neutrons'

arrivals  $\dots, T_{k-1}, T_k, T_{k+1}, \dots$  are independent and exponentially distributed with expectation  $1/\lambda$ . An alternative and equivalent description is that  $X_0(t)$  is a stochastic process such that

- (i)  $X_0(0) = 0$ ,
- (ii)  $X_0(t_2) - X_0(t_1) \in Po(\lambda_0(t_2 - t_1))$ , if  $0 \leq t_1 < t_2$ ,
- (iii)  $X_0(t_2) - X_0(t_1)$  and  $X_0(t_4) - X_0(t_3)$  are independent if  $0 \leq t_1 < t_2 \leq t_3 < t_4$ .

The parameter  $\lambda_0$  is the intensity of the process  $X_0(t)$ , which is unknown and will be estimated.

At the layer each neutron is absorbed with the certain probability  $p_0$  (absorption efficiency). The probability of absorption  $p_0$  is also an unknown parameter, its dependence on the wavelength  $\mu_0$  of the incident neutron (2012, D. Anevski and R. Hall-Wilton) is however on a known functional form. This property will be used to make inference about the parameter  $\mu_0$ .

The paper is organized as follows: Section 2 provides the general scheme of the neutron detector and the modeling of neutron interactions with the detector layers. Section 3 is devoted to the inference of the parameters: We derive the ML estimators for the intensity of an incident beam  $\lambda_0$  and absorption efficiency  $p_0$ , in Lemma 2 and 3 we discuss the uniqueness of solutions to the score equations, and in Theorem 1, which is one of the main result of this paper, we derive the consistency and asymptotic normality of the ML estimators. Studying the formulas for the asymptotic variance, we get as a corollary in Lemma 4 that the variance of the MLE of the absorption probability is a decreasing function of the number of layers in the neutron detector. Finally in Corollary 1 and 2, we derive the consistency and asymptotic normality of the MLE of the wavelength of the neutron beam  $\mu_0$ . Using these final results we are able to construct confidence intervals for the wavelength. Section 4 gives simulation study to explore the estimators performance. Section 5 ends with the discussion of the results presented in the paper and plans for future work. Proofs of all results are given in an Appendix.

## 2 Scheme of a discrete spacing detector

Assume that an incident beam of neutrons hits the first layer of the detector, cf. Fig.1. At the layer a neutron can possibly be absorbed and detected. If a neutron is not absorbed it will go through the detector layer. We assume that these are the only two possibilities for the neutron interaction with a layer. Let  $p_0$  be the probability of an absorption of a neutron, so that  $1 - p_0$

is the probability of its transmission. If a neutron is absorbed, it will then be detected. Let  $X_1(t)$  be the number of neutrons that are absorbed at the first layer, so that  $X_1^{tr}(t) = X_0(t) - X_1(t)$  is the number of transmitted neutrons.

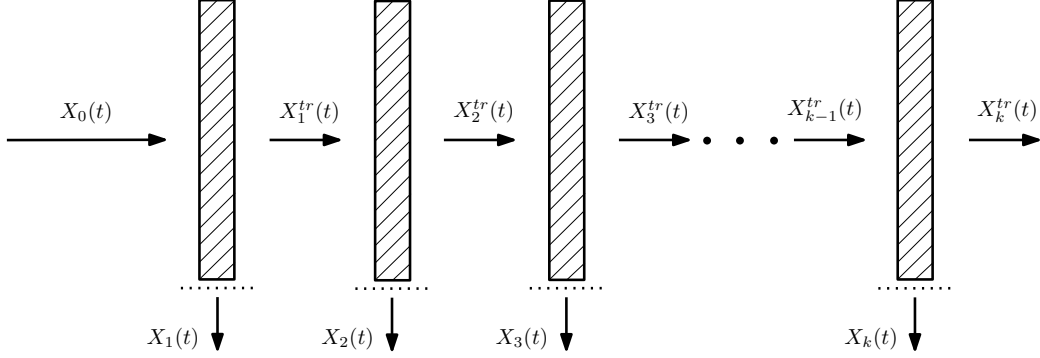


Figure 1: The scheme of the detector.

Now assume that the transmitted beam of neutrons  $X_1^{tr}(t)$  hits the next layer, at which again each neutron can either be absorbed (with the same probability  $p_0$  as at the previous one) and then detected or transmitted again. Let  $X_2(t)$  be the number of neutrons absorbed at the second layer and let  $X_2^{tr}(t) = X_1^{tr}(t) - X_2(t)$  be the number of transmitted neutrons. We assume that the registrations (absorptions) of different particles are independent and the times of absorption and travelling from layer to layer are negligibly small. This behaviour is repeated at each layer and gives the general scheme for the neutron beams absorption and transmission in the detector.

Let  $X_i(t)$  be the number of neutrons absorbed at the layer  $i$  in the time interval  $[0, t]$  and let  $X_i^{tr}(t)$  be the number of transmitted neutrons in the same time interval, through the layer  $i$ ,  $i = 1, \dots, k$ . Then  $X_i(t)$  and  $X_i^{tr}(t)$  are counting processes and

$$\begin{aligned} X_i(0) &= 0, \\ X_i^{tr}(0) &= 0, \end{aligned}$$

for  $i = 1, \dots, k$ .

The next lemma shows that  $\{X_i(t)\}_{i \geq 1}$  are jointly independent Poisson processes with parameters  $\lambda p_0(1 - p_0)^{i-1}$ , respectively.

**Lemma 1** *Assume that the detector consists of  $k \leq \infty$  layers (with  $k = \infty$  as a possibility), that are equally spaced at the depths  $d_1 = d, d_2 = 2d, \dots, d_k = kd$  (Fig.1). Assume that  $X_0(t)$  is a Poisson processes with intensity  $\lambda_0$  and*

the probabilities of absorption are the same for each layer  $p_0(d_i) \equiv p_0, i \geq 1$ .  
Let

$$\begin{aligned} X_i^{tr}(t) &:= X^{tr}(t, id), \\ X_i(t) &:= X(t, id), \end{aligned}$$

for  $i = 1, \dots, k$ , be the number of transmitted and detected neutrons at the layer  $i$  and time  $t$ , where  $X_i^{tr}(t)$  has intensity  $(1 - p_0)^i \lambda_0$ ,  $X_i(t)$  has  $p_0(1 - p_0)^{i-1} \lambda_0$ . Then  $\{X_i(t)\}$  are jointly independent Poisson processes.

The proof is given in the Appendix.

### 3 Inference for the parameters

Now suppose that we have run an experiment at the neutron detector, the result of which is a sequence of counts of the number of detected neutrons along the detector. Let us denote the data as a vector  $x = (x_1, \dots, x_k)$  of integers, with  $x_i$  the number of observed neutrons at detector layer  $i$ . From Lemma 1 we know that these are observations of independent Poisson distributed r.v.'s, with unknown expectation  $p_0(1 - p_0)^{i-1} \lambda_0$ , for the observation at layer  $i$ . Using these data we would like to estimate the unknown parameters  $p_0$  and  $\lambda_0$ .

We can think of two feasible general approaches for inference in this model: either a likelihood approach or a regression approach. It turns out that the least squares regression approach will give us polynomial equations that become intractable. We will use Maximum Likelihood estimation for making inference about the parameters  $(p_0, \lambda_0)$ .

#### 3.1 The ML estimator of the probability of absorption $p_0$ and the intensity of an incident beam $\lambda_0$

We are interested in deriving consistency and asymptotic normality of the estimators. For this we need to describe what we mean by letting "the amount of data" go to infinity. There are several ways to model this. We can either let the experiment time  $t$  increase, or we can view the problem as a repeated measurement problem and thus make several,  $n$  of them, independent measurements during a fixed time interval  $[0, t]$  and let  $n$  go to infinity. If the model for the process used is that of a Poisson process then the two approaches will give quantitative the same limit results. We will use the second

approach. Thus we view the problem as a repeated sample problem. We assume that during  $n$  fixed equal time intervals  $[0, t]$  and using beams with the same fixed intensity  $\lambda$ , we perform  $n$  repeated experiments with  $k$  data points in each.

The inference problem can be described as follows: We perform  $n$  experiments. For each experiment  $i = 1, \dots, n$ , we measure the number of neutrons  $X_{ij}$  detected at layer  $j = 1, \dots, k$  during the time interval  $[0, t]$ . Thus  $\{X_{ij}\}_{i,j=1}^n$  are random variables and  $\{x_{ij}\}_{i,j=1}^n$  are the values which  $X_{ij}$  take in the  $n$  experimental runs. Let  $(p, \lambda)$  denote the parameters, that are assumed to lie in the parameter space  $[0, 1] \times [0, \infty)$ . Let  $(p_0, \lambda_0)$  denote true and unknown parameter values in the distribution of the data. Introduce the vectors  $\mathbf{X}_j = (X_{1j}, \dots, X_{kj})^T$  and  $\mathbf{x}_j = (x_{1j}, \dots, x_{kj})^T$ , respectively. Note that the vectors  $\mathbf{X}_j$  are independent random vectors with jointly independent components  $X_{ij}$ , by Lemma 1, from  $n$  independent experiment rounds. Finally denote  $\mathbf{X} = [\mathbf{X}_1, \dots, \mathbf{X}_n]$  and  $\mathbf{x} = [\mathbf{x}_1, \dots, \mathbf{x}_n]$ , and note that these are  $k \times n$  matrices of discrete random variables and of integers values, respectively.

Thus we let  $X_{ij}$  be the number of neutrons observed at the layer  $i$  at the experiment round  $j$  with probability mass function

$$f(x_{ij}|p, \lambda) = e^{-m_i} \frac{m_i^{x_{ij}}}{x_{ij}!},$$

where  $m_i = p(1-p)^{i-1}\lambda t$ . Then each vector  $\mathbf{X}_j = (X_{1j}, \dots, X_{kj})^T$  has the joint distribution

$$f(\mathbf{x}_j|p, \lambda) = \prod_{i=1}^k f(x_{ij}|p, \lambda) = \prod_{i=1}^k e^{-m_i} \frac{m_i^{x_{ij}}}{x_{ij}!}.$$

Assume that  $k > 1$ . The likelihood is given by

$$L_n(p, \lambda|\mathbf{x}) = \prod_{j=1}^n f(\mathbf{x}_j|p, \lambda) = \prod_{j=1}^n \prod_{i=1}^k e^{-m_i} \frac{m_i^{x_{ij}}}{x_{ij}!} \quad (1)$$

and the log-likelihood is

$$\begin{aligned} l_n(p, \lambda|\mathbf{x}) &= \log L_n(p, \lambda|\mathbf{x}) \\ &= \sum_{j=1}^n g\left\{-\sum_{i=1}^k m_i + \sum_{i=1}^k x_{ij} \log m_i - \sum_{i=1}^k \log x_{ij}! g\right\}. \end{aligned} \quad (2)$$

The score functions are

$$\begin{cases} \Psi_{n,1}(p, \lambda|\mathbf{x}) = \frac{1}{n} \frac{\partial l_n}{\partial \lambda} = -t \sum_{i=1}^k p(1-p)^{i-1} + \frac{1}{n\lambda} \sum_{j=1}^n \sum_{i=1}^k x_{ij}, \\ \Psi_{n,2}(p, \lambda|\mathbf{x}) = \frac{1}{n} \frac{\partial l_n}{\partial p} = -\lambda t \sum_{i=1}^k \{(1-p)^{i-1} - p(i-1)(1-p)^{i-2}\} \\ + \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^k x_{ij} \left( \frac{1}{p} - \frac{i-1}{1-p} \right), \end{cases}$$

which can be simplified to

$$\begin{cases} \Psi_{n,1}(p, \lambda|\mathbf{x}) = \frac{\bar{s}_n - \lambda t(1-(1-p)^k)}{\lambda}, \\ \Psi_{n,2}(p, \lambda|\mathbf{x}) = \frac{(1-p)(\bar{s}_n + \bar{z}_n) - \bar{z}_n - \lambda t(k(1-p)^k - k(1-p)^{k+1})}{p(1-p)}, \end{cases} \quad (3)$$

where

$$\bar{s}_n = \frac{1}{n} \sum_{j=1}^n s_j, \quad s_j = \sum_{i=1}^k x_{ij}, \quad \bar{z}_n = \frac{1}{n} \sum_{j=1}^n z_j, \quad z_j = \sum_{i=1}^k (i-1)x_{ij}. \quad (4)$$

The ML estimator  $(\hat{p}_n, \hat{\lambda}_n)$  is the solution to the likelihood equations

$$\begin{cases} \Psi_{n,1}(p, \lambda|\mathbf{x})|_{p=\hat{p}_n, \lambda=\hat{\lambda}_n} = 0, \\ \Psi_{n,2}(p, \lambda|\mathbf{x})|_{p=\hat{p}_n, \lambda=\hat{\lambda}_n} = 0. \end{cases} \quad (5)$$

If we assume that  $\hat{p}_n(1 - \hat{p}_n) \neq 0$ ,  $\hat{\lambda}_n \neq 0$  we get the system of equations

$$\begin{aligned} & \begin{cases} \bar{s}_n - \hat{\lambda}_n t(1 - (1 - \hat{p}_n)^k) = 0, \\ (1 - \hat{p}_n)(\bar{s}_n + \bar{z}_n) - \bar{z}_n - \hat{\lambda}_n t(k(1 - \hat{p}_n)^k - k(1 - \hat{p}_n)^{k+1}) = 0. \end{cases} \\ \Leftrightarrow & \begin{cases} \bar{s}_n - \hat{\lambda}_n t(1 - \hat{y}_n^k) = 0, \\ a_n \hat{y}_n^{k+1} - b_n \hat{y}_n^k + c_n \hat{y}_n - d_n = 0, \end{cases} \end{aligned} \quad (6)$$

where

$$\begin{aligned} a_n &= -\bar{s}_n - \bar{z}_n + k\bar{s}_n, \\ b_n &= -\bar{z}_n + k\bar{s}_n, \\ c_n &= \bar{z}_n + \bar{s}_n, \\ d_n &= \bar{z}_n, \\ \hat{y}_n &= 1 - \hat{p}_n. \end{aligned} \quad (7)$$

Thus the ML estimator  $(\hat{p}_n, \hat{\lambda}_n)$  is the solution to the system of equations (6). Obviously this system has exactly one solution  $(\hat{p}_n, \hat{\lambda}_n)$  if and only if its second equation has exactly one root. The next result gives a necessary and sufficient condition for this, in terms of the coefficients of the second equation.

**Lemma 2** *The function*

$$\tilde{f}(y) = a_n y^{k+1} - b_n y^k + c_n y - d_n,$$

*with coefficients given in (7), has one zero in the open interval  $(0, 1)$  when the inflection point  $y_{i.p.}$  satisfies the inequality*

$$y_{i.p.} := \frac{b_n(k-1)}{a_n(k+1)} < 1, \quad (8)$$

*and no zeros in  $(0, 1)$  when  $y_{i.p.} \geq 1$ .*

The proof is given in the Appendix.

From Lemma 2, the condition of existence and uniqueness of ML estimator  $(\hat{p}_n, \hat{\lambda}_n)$  is

$$\frac{b_n(k-1)}{a_n(k+1)} < 1,$$

but there is no guarantee that it holds for a finite  $n$ . However, we can show that this holds infinitely often, almost surely.

**Lemma 3** *Let  $A_n = \{\text{Equation (6) has a root in } (0, 1)\}$ . Then  $A_n$  occurs infinitely often, almost surely.*

The proof is given in the Appendix.

### 3.1.1 Asymptotic properties of the ML estimator

**Theorem 1** *Let  $\mathbf{X}$  be a  $k \times n$  matrix, where  $X_{ij}$  are  $k \times n$  jointly independent Poisson random variables with densities given by*

$$f(x_{ij}|\boldsymbol{\theta}) = e^{-m_i} \frac{m_i^{x_{ij}}}{x_{ij}!}, \quad i = 1, \dots, k, \quad j = 1, \dots, n, \quad x_{ij} \in \mathbb{Z}^+,$$

*where  $m_i = p(1-p)^{i-1}\lambda t$ ,  $\boldsymbol{\theta} = (p, \lambda)$ , and the true parameters  $\boldsymbol{\theta}_0 = (p_0, \lambda_0) \in \Omega = \{\boldsymbol{\theta} : 0 < p < 1, 0 < \lambda < \infty\}$ .*

*Then the ML estimator  $\hat{\boldsymbol{\theta}}_n = (\hat{p}_n, \hat{\lambda}_n)$ , given in (5), of the parameters  $\boldsymbol{\theta}_0 = (p_0, \lambda_0)$  is consistent*

$$\hat{\boldsymbol{\theta}}_n \xrightarrow{p} \boldsymbol{\theta}_0$$

*and asymptotically normal*

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}g(\mathbf{0}, [\mathbf{I}(\boldsymbol{\theta}_0)]^{-1}g),$$



where  $\mathbf{I}(\boldsymbol{\theta})$  is the information matrix

$$\mathbf{I}(\boldsymbol{\theta}) = \frac{1}{k} \sum_{i=1}^k \mathbf{I}_{(i)}(\boldsymbol{\theta}), \quad (9)$$

where  $\mathbf{I}_{(i)}(\boldsymbol{\theta})$  denotes the information matrix corresponding to  $f(x_{ij}|\boldsymbol{\theta})$  with fixed  $i$  and has elements

$$\begin{aligned} [\mathbf{I}_{(i)}(\boldsymbol{\theta})]_{\alpha\beta} &= \mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial}{\partial \theta_{\alpha}} \log f(X_{ij}|\boldsymbol{\theta}) \cdot \frac{\partial}{\partial \theta_{\beta}} \log f(X_{ij}|\boldsymbol{\theta}) \right] \\ &= \mathbb{E}_{\boldsymbol{\theta}} \left[ -\frac{\partial^2}{\partial \theta_{\alpha} \partial \theta_{\beta}} \log f(X_{ij}|\boldsymbol{\theta}) \right], \end{aligned} \quad (10)$$

for  $\alpha, \beta = 1, 2$ .

Note:  $X_{ij} \stackrel{d}{=} X_{il}$  for all  $i = 1, \dots, k$  and  $j, l = 1, \dots, n$ .

The proof is given in the Appendix.

From (9) in the Theorem above, and using (10), we get after simplification

$$\begin{aligned} [\mathbf{I}(\boldsymbol{\theta})]_{pp} &= \lambda t \frac{h(p)}{(1-p)p^2}, \\ [\mathbf{I}(\boldsymbol{\theta})]_{\lambda p} &= tk \frac{(1-p)^{k-1} - (1-p)^k}{p}, \\ [\mathbf{I}(\boldsymbol{\theta})]_{\lambda\lambda} &= t \frac{1 - (1-p)^k}{\lambda}, \end{aligned} \quad (11)$$

where

$$h(p) = 1 - k^2(1-p)^{k+1} + (2k^2 - 1)(1-p)^k - k^2(1-p)^{k-1}.$$

This implies that for the asymptotic variances we have the following asymptotic expressions

$$\begin{aligned} \sigma_{\hat{p}}^2(p_0, \lambda_0) = [\mathbf{I}(\boldsymbol{\theta}_0)]_{pp}^{-1} &= \frac{(1 - (1-p_0)^k)(1-p_0)p_0^2}{\lambda_0 t q(p_0)} \rightarrow \frac{(1-p_0)p_0^2}{\lambda_0 t}, \\ \sigma_{\hat{\lambda}}^2(p_0, \lambda_0) = [\mathbf{I}(\boldsymbol{\theta}_0)]_{\lambda\lambda}^{-1} &= \frac{\lambda h(p_0)}{t q(p_0)} \rightarrow \frac{\lambda_0}{t}, \\ \sigma_{\hat{p}, \hat{\lambda}}^2(p_0, \lambda_0) = [\mathbf{I}(\boldsymbol{\theta}_0)]_{\lambda p}^{-1} &= \frac{k p_0 ((1-p_0)^k - (1-p_0)^{k-1})}{t q(p_0)} \rightarrow 0, \end{aligned} \quad (12)$$

as  $k \rightarrow \infty$ , where

$$q(p) = (1-p)^{2k} - k^2(1-p)^{k+1} + 2(k^2-1)(1-p)^k - k^2(1-p)^{k-1} + 1. \quad (13)$$

We are mainly interested in estimation of  $p_0$ , since there is a functional relation between it and the wavelength  $\mu_0$  of the incident neutrons, cf. (15) and (16) below. Therefore we analyse the behaviour of  $\sigma_{\hat{p}}^2(p, \lambda)$ . It can be shown that  $\sigma_{\hat{p}}^2(p, \lambda)$  is a strictly decreasing function of  $k$  for every  $p \in (0, 1)$ .

**Lemma 4**  $\sigma_{\hat{p}}(p, \lambda)$  is strictly decreasing as a function of  $k$ , on  $k > 1$ , for all  $p \in (0, 1)$ .

The proof is given in the Appendix.

Figure 2 is a plot of the standard deviation  $\sigma_{\hat{p}}(p, \lambda)$  for  $\hat{p}$  as a function of the number of layers  $k$  and of the absorption probability  $p_0$ . In particular we note that  $\sigma_{\hat{p}}$  strongly depends on the absorption efficiency  $p_0$  for a small number of layers in the detector ( $k < 10$ ).

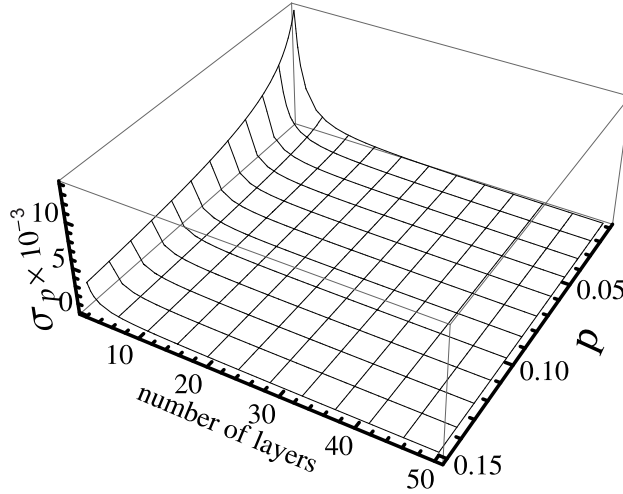


Figure 2: The dependence of the standard deviation  $\sigma_{\hat{p}}$  on the number of layers in the detector;  $\lambda_0 = 10^5$ ,  $t = 1$  s.

### 3.2 Estimation of the wavelength $\mu_0$ of an incident beam.

From the Theorem 1 the ML estimator  $\hat{p}$  of  $p_0$  satisfies

$$\sqrt{n}(\hat{p}_n - p_0) \xrightarrow{d} \mathcal{N}g(0, \sigma_{\hat{p}}^2(p_0, \lambda_0)g), \quad (14)$$

as  $n \rightarrow \infty$ .

We are interested in estimating the wavelength, for a monochromator neutron beam. The probability of absorption  $p$  depends on the neutron wavelength  $\mu$  as (2012, D. Anevski and R. Hall-Wilton)

$$p = 1 - e^{-\Sigma(\mu)\rho_{at}d_l}, \quad (15)$$

where  $\Sigma(\mu)$  is the cross-section of absorption in the matter,  $\rho_{at} = 10^{29} \text{ m}^{-3}$  is the atomic density of  $^{10}\text{B}$  the  $B_4C$  coating,  $d_l = 10^{-6} \text{ m}$  is the thickness of the layer. The neutron cross-section  $\Sigma(\mu)$  can be modelled as a linear function,

$$\Sigma(\mu) = \varsigma\mu, \quad (16)$$

where the coefficient  $\varsigma$  is different for different materials (2012, D. Anevski and R. Hall-Wilton). This coefficient does not depend on the neutron wavelength and has been measured experimentally (1959, H. W. Schmitt et al.). From the results in the paper by Schmitt we conclude that the estimator  $\hat{\varsigma}$  of  $\varsigma_0$  is unbiased and asymptotically normal

$$\sqrt{n'}(\hat{\varsigma}_{n'} - \varsigma_0) \xrightarrow{d} \mathcal{N}g(0, \sigma_{\hat{\varsigma}}^2 g), \text{ as } n' \rightarrow \infty.$$

Here  $n'$  is the number of experimental runs (measurements) performed in an experiment such as in (1959, H. W. Schmitt et al.) to estimate  $\varsigma$ ,  $\varsigma_0$  is the true value and  $\sigma_{\hat{\varsigma}}^2$  is its asymptotic variance.

Let us rewrite (15) as

$$p = 1 - e^{-\chi\mu}, \quad (17)$$

where

$$\chi = \rho_{at}d_l\varsigma, \quad (18)$$

The plug-in estimator  $\hat{\chi} = \rho_{at}d_l\hat{\varsigma}$  of  $\chi$  is then obviously asymptotically normal

$$\sqrt{n'}(\hat{\chi}_{n'} - \chi_0) \xrightarrow{d} \mathcal{N}g(0, \sigma_{\hat{\chi}}^2 g), \quad (19)$$

where  $\chi_0 = \rho_{at} d_l \varsigma_0$  and  $\sigma_\chi^2 = \rho_{at}^2 d_l^2 \sigma_\varsigma^2$ .

Solving for  $\mu$  in (17), we get

$$\mu(p, \chi) = -\frac{\log(1-p)}{\chi}. \quad (20)$$

In the next result we combine the two limit distribution results, for  $\hat{p}_n$  and for  $\hat{\chi}_n$  to get a limit distribution result for the plug-in estimator for  $\mu_0$  obtained from the relation (20). In order to formalize this is a proper way, we introduce a factor  $\gamma$ , which is merely the (asymptotic) ratio between  $n'$  and  $n$ . The result will in practical finite-sample situation be used in exactly that way: by letting  $\gamma = n'/n$  and use the limit distribution to provide asymptotic confidence intervals or tests.

**Corollary 1** *The plug-in estimator  $\hat{\mu} = \mu(\hat{p}_n, \hat{\chi}_{n'})$  of  $\mu_0 = \mu(p_0, \chi_0)$  is asymptotically normal*

$$\sqrt{n}(\mu(\hat{p}_n, \hat{\chi}_{n'}) - \mu(p_0, \chi_0)) \xrightarrow{d} \mathcal{N}(0, \sigma_\mu^2), \quad (21)$$

where

$$\sigma_\mu^2 = \left[ \frac{\partial \mu}{\partial p}(p_0, \chi_0) \right]^2 \sigma_p^2(p_0, \lambda_0) + \frac{1}{\gamma} \left[ \frac{\partial \mu}{\partial \chi}(p_0, \chi_0) \right]^2 \sigma_\chi^2$$

as  $n \rightarrow \infty$ , where  $n$  is the number of measurements for  $\hat{p}_n$  and  $n' = \lceil \gamma n \rceil$ ,  $\gamma > 0$ , is the number of measurement for  $\hat{\chi}_{n'}$  ( $\lceil \gamma n \rceil$  is smallest integer not less than  $\gamma n$ ).

The proof is given in the Appendix.

From Corollary 1 it follows that the asymptotic variance  $\sigma_\mu^2$  is a function of the true values of  $p$ ,  $\lambda$  and  $\chi$ , and also of the variance of  $\chi$ , which are all unknown. In order to construct the confidence interval for  $\mu_0$ , we will use estimates of  $p$ ,  $\lambda$  and  $\chi$ , and a pooled variance estimate for  $\chi$ , to give a plug-in estimate of  $\sigma_\mu^2$ .

Introduce the notation

$$S_n^2 = \left[ \frac{\partial \mu}{\partial p}(\hat{p}_n, \hat{\chi}_{n'}) \right]^2 \sigma_p^2(\hat{p}_n, \hat{\lambda}_n) + \frac{1}{\gamma} \left[ \frac{\partial \mu}{\partial \chi}(\hat{p}_n, \hat{\chi}_{n'}) \right]^2 \hat{\sigma}_\chi^2, \quad (22)$$

where both the sample mean  $\hat{\chi}_{n'}$  and the sample variance  $\hat{\sigma}_\chi^2$  are based on  $n'$  measurements, and  $(\hat{p}_n, \hat{\lambda}_n)$  are the ML estimators of  $(p_0, \lambda_0)$  based on  $n$  measurements.

## Corollary 2

$$\frac{\sqrt{n}(\mu(\hat{p}_n, \hat{\chi}_{n'}) - \mu(p_0, \chi_0))}{S_n} \xrightarrow{d} \mathcal{N}(0, 1), \quad (23)$$

as  $n \rightarrow \infty$ , where  $n' = \lceil \gamma n \rceil$ ,  $\gamma > 0$  and  $S_n$  is given by (22).

The proof is given in the Appendix.

Using the above limit distribution result for the ML estimator  $\hat{\mu}$  of  $\mu_0$  we can construct approximate confidence interval for  $\mu_0$ . The approximate 100(1 -  $\alpha$ ) percent confidence interval on  $\mu_0$  is

$$[\mu(\hat{p}_n, \hat{\chi}_{n'}) - Z_{\alpha/2} \frac{S_n}{\sqrt{n}}, \mu(\hat{p}_n, \hat{\chi}_{n'}) + Z_{\alpha/2} \frac{S_n}{\sqrt{n}}], \quad (24)$$

where  $Z_{\alpha/2}$  is the upper  $\alpha/2$  percentage point of the standard normal distribution.

## 4 Simulation experiment

Our main goal in this paper is to estimate the wavelength  $\mu_0$  of the incident neutron beam. In the previous sections we have derived the ML estimator of the unknown wavelength as well as given a formula for a confidence interval with approximate (asymptotically correct) coverage probability. In this section we perform simulation experiments to evaluate the estimators performance. In particular we illustrate the dependence of the individual terms in variance on the number of layers (Figure 3), the confidence interval widths dependence on the number of incoming neutrons (or equivalently the time length of the experiment) (Figure 4), and the confidence interval widths dependence on the number of layers (Figure 5). All simulations are done in Wolfram Mathematica 9.

We simulate a Poisson process  $X_0(t)$  a number of times  $n$ , for  $n = 10, 100$ , for the parameters values  $p_0 = 0.05, 0.07, 0.1$ ,  $\lambda_0 = 10^5 \text{ s}^{-1}$  and  $t = 1 \text{ s}$ . The ML estimators  $(\hat{p}_n, \hat{\lambda}_n)$  are calculated on the simulated data. We recall the relation between  $\chi$  and  $\varsigma$  in (18), and note that  $\rho_{at}$  and  $d_l$  are known. The estimator of  $\varsigma$  is assumed to be asymptotically normal, with mean value the sample mean and variance equal to a pooled variance estimate using three series of 15 measurements, which gives in total  $n' = 45$  experimental data point, cf. (1959, H. W. Schmitt et al.).

First, we analyse the dependence of the confidence interval width on the number of detectors layers. Let us rewrite the expression for  $\frac{S_n^2}{n}$  as

$$\frac{S_n^2}{n} = (S_{\hat{\mu}}^{(p)})^2 + (S_{\hat{\mu}}^{(\chi)})^2,$$

where

$$\begin{aligned} (S_{\hat{\mu}}^{(p)})^2 &= \frac{1}{n} \left[ \frac{\partial \mu}{\partial p}(p_0, \chi_0) \right]^2 \sigma_{\hat{p}}^2(p_0, \lambda_0) \\ &= \frac{\sigma_{\hat{p}}^2(p_0, \lambda_0)}{n(1-p_0)^2 \chi_0^2}, \end{aligned}$$

$$\begin{aligned} (S_{\hat{\mu}}^{(\chi)})^2 &= \frac{\gamma}{n} \left[ \frac{\partial \mu}{\partial \chi}(p_0, \chi_0) \right]^2 \sigma_{\hat{\chi}}^2 \\ &= \frac{\sigma_{\hat{\chi}}^2 \log^2(1-p_0)}{n' \chi_0^4}. \end{aligned}$$

The first term  $S_{\hat{\mu}}^{(p)}$  constitutes the contribution to the total variance from the ML estimators  $(\hat{p}_n, \hat{\lambda}_n)$  of  $(p, \lambda)$  whereas the second term  $S_{\hat{\mu}}^{(\chi)}$  is the contribution from the estimator  $\hat{\chi}_n$  of  $\chi$ . For the purpose of the experiments we perform, we can view the second term as a kind of systematic error, outside of our control.

Figure 3 shows the dependence of  $S_{\hat{\mu}}^{(p)}$  and  $S_{\hat{\mu}}^{(\chi)}$  on the number of the layers in the detector for 10 and 100 runs of the experiment. We note in particular that  $S_{\hat{\mu}}^{(p)}$  and  $S_{\hat{\mu}}^{(\chi)}$  are of the same size at  $k \approx 25$  for  $n = 10$  experimental runs and at  $k \approx 15$  for  $n = 100$ .

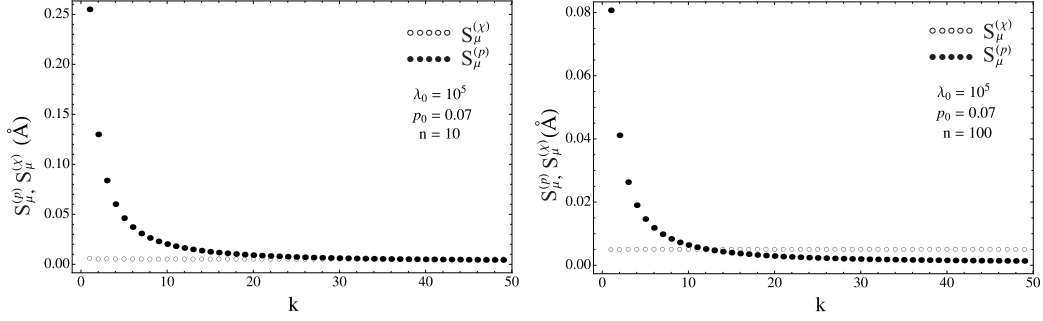


Figure 3: The dependence of  $S_{\hat{\mu}}^{(p)}$  and  $S_{\hat{\mu}}^{(\chi)}$  on the number of layers  $k$ .

Next, we analyse the dependence of the confidence interval width, which is equal to  $2Z_{\alpha/2} \frac{S_n}{\sqrt{n}}$ , on the total number of incoming neutrons  $n\lambda t$ . In Figure 4 we have plotted the 99 % confidence interval width, for  $p_0 = 0.07$  ( $\approx 3.4$  (Å)) and  $k = 30$  layers, as a function of the total number of incoming neutrons.

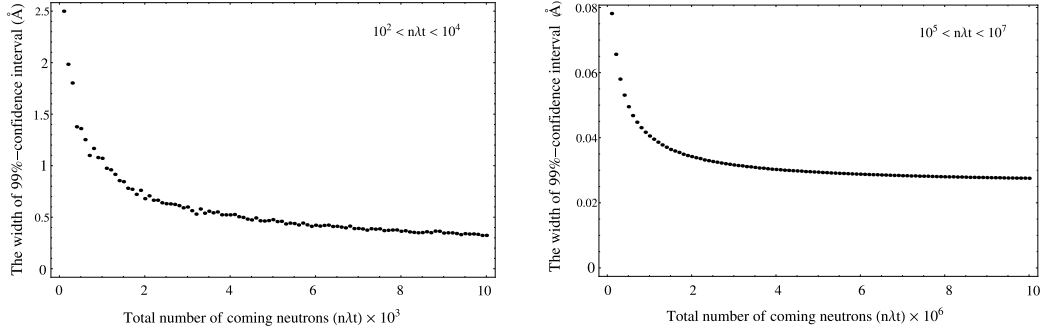


Figure 4: 99 % confidence interval width for ( $p_0 = 0.07$ ,  $k = 30$ ): low statistics  $10^2 < n\lambda t < 10^4$  (left), high statistics  $10^5 < n\lambda t < 10^7$  (right).

In Figure 5 we have plotted confidence interval bars as a function of the number of layers, for  $p_0 = 0.05, 0.07, 0.1$  (which correspond to the wavelengths  $\mu_0 = 2.4, 3.4$  and  $4.9$  (Å)) and  $n = 10, 100$ .

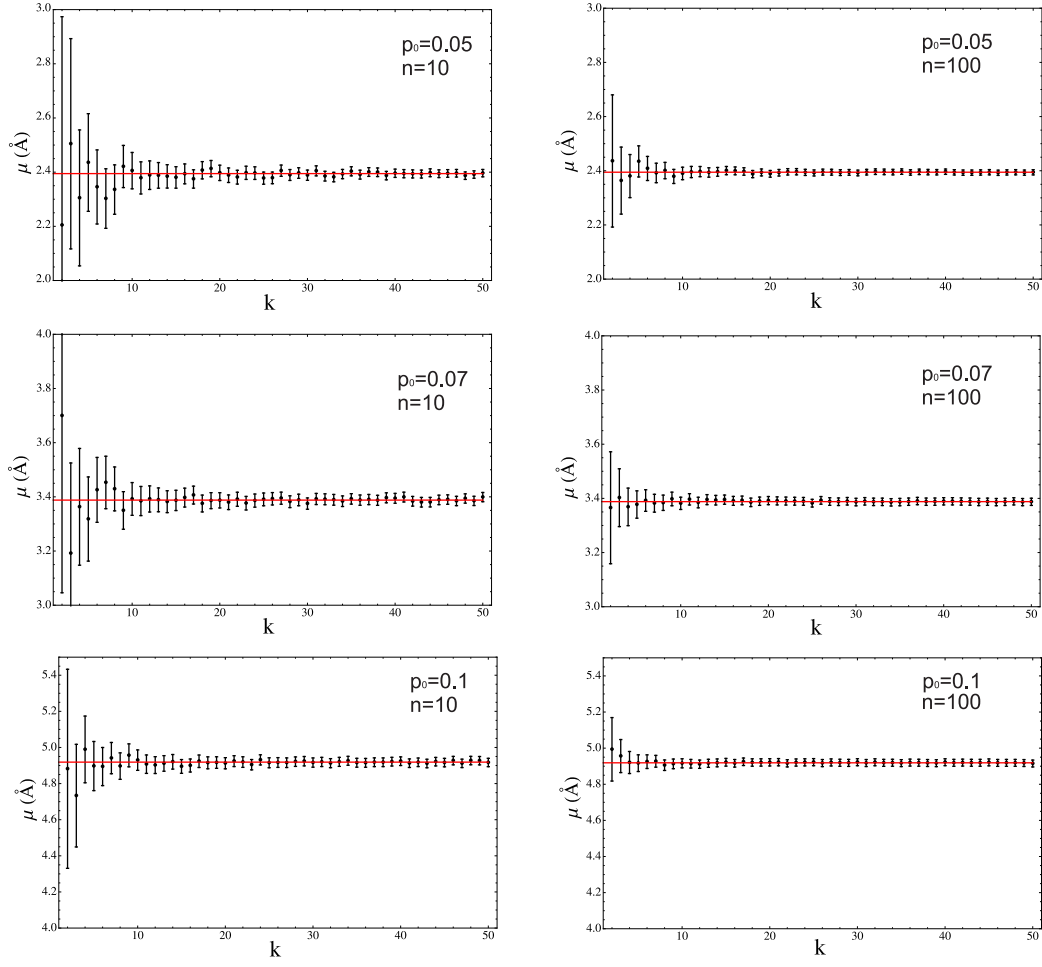


Figure 5: 99% confidence interval for  $\mu_0$  based on simulations for  $n = 10, 100$  and  $p_0 = 0.05, 0.07, 0.1$ ,  $\lambda_0 = 10^5$ ,  $t = 1s$ . The red line is the true value of  $\mu_0$ .



## 5 Conclusions

The results here show that it is statistically possible to determine neutron energy, for a monochromatic beam, with good precision from multilayer neutron detectors. With relatively few layers ( $\leq 10$ ), already maximal information can be extracted and many more layers do not statistically significantly improve the precision of the results.

In the limit of high statistics ( $> 10^7$  neutrons), a statistical precision (width of 99 % confidence interval) less than 0.1 ( $\text{\AA}$ ) on the determination of the wavelength of the beam in the range 2.5-5 ( $\text{\AA}$ ) is possible (Fig.5). Uncertainty on the neutron cross section of the Boron-10 isotope becomes dominant in the regime of high statistics and more than 10-20 layers, and a systematic limit on the determination of the wavelength. This means again that more than 10-20 layers are not needed (Fig.4).

An interesting further outcome of this, is that it might be possible, in high statistics experiments, with a precisely determined wavelength monochromatic neutron beam, to improve the statistical measurement of the Boron-10 cross section, by using an inverse of the method described in this manuscript. The systematic effects of such a measurement might be significant.

In the limit of low statistics, a precision of 1( $\text{\AA}$ ) in determining the wavelength of the monochromatic neutron beam is still possible. (new figure).

In a real detector there may be a degradation in the result achieved coming from systematic effects resulting from defects in the detector.

Modelling the neutron beam as a Poisson process, with unknown arrival intensity, of particles, all with the same and unknown wavelength, we defined the Maximum Likelihood estimator of the two unknown parameters, and showed that asymptotically, almost surely, the MLE exists (Lemma 3). The asymptotic normality of the MLE was established (Theorem 1), with analytic expressions for the asymptotic variance (see also Lemma 4), making it possible to make statistical tests and to construct confidence intervals. The limit distribution result was next combined with a limit distribution result for a parameter in the functional relation between the wavelength and absorption probability, to finally derive a limit distribution result for the MLE of the unknown wavelength of the neutrons (Corollary 2). The asymptotic variance of the wavelength estimate is seen to be a decreasing function of the total number of the incoming neutrons, as well a decreasing function of the number of layers in the detector (formula (22) and Lemma 4). These two last facts enabled us to draw conclusions about the precision in the estimate the unknown wavelength (with precision meaning the width of a confidence interval for the wavelength), and how that precision is affected by the detector construction, i.e. the number of layers of the detector, as well as by the

run time of the experiment, i.e. the total number of neutrons that hit the first layer of the detector.

We conclude by commenting that this manuscript concentrated on a monochromatic neutron beam; in future this will be generalised to discrete and continuous wavelength distributions of neutrons and real-world use cases thereof.

## 6 Acknowledgements

VP's research is fully supported by the Swedish Research Council (VR). The research of DA, RHW and KK is partially supported by the VR. The authors gratefully acknowledge the VR's support.

Author's present address: Solvegatan 18, 223 62 Lund

E-mail: [pastuhov@maths.lth.se](mailto:pastuhov@maths.lth.se)

## 7 Appendix

We first give a short proof of Lemma 1 based on Bernoulli splitting of a Poisson process.

*Short proof of Lemma 1.* Let the number of incident neutrons in the time interval  $[0, t]$  be equal  $n$ , i.e.  $X_0(t) = n$ . Then the conditional distribution of  $X_i(t)$  is given by

$$(X_1(t), \dots, X_k(t), X_k^{tr}(t) | X_0 = n) \in Mult(n, q_1, q_2, \dots, q_k, q_k^{tr}).$$

where  $q_i = p_0(1 - p_0)^{i-1}$ ,  $q_k^{tr} = (1 - p_0)^k$ . The statement of the Lemma now follows from the property of a Bernoulli Splitting of a Poisson process, c.f. Theorem 5.17 in (2009, V. G. Kulkarni).  $\square$

For completeness we give full, slightly different, proof of Lemma 1 .

*Proof of Lemma 1 .* First, we show that  $\{X_i\}$  are jointly independent Poisson processes. We prove this by using induction on the layer number  $i$ .

(1) : ( $i = 1$ ) We first show

(a)  $X_1, X_2$  are independent,

(b)  $X_1$  is independent of  $X_1^{tr}$ .

Since,  $X_1, X_1^{tr}$  are obtained by thinning of the incoming Poisson process  $X_0$ , with thinning probabilities  $p, 1 - p$  respectively, the processes  $X_1, X_1^{tr}$  are independent Poisson processes, with intensities  $\lambda p, \lambda(1 - p)$ , respectively, and thus part (b) holds.

To show part (a), the transmitted and absorbed processes at layer 2,  $X_2, X_2^{tr}$ , are obtained by thinning of the Poisson process  $X_1^{tr}$ , which, since  $X_1$  and  $X_1^{tr}$  are independent and since the classification mechanism into absorbed or transmitted neutrons at layer 2 is independent of what goes on at layer 1, we have that the vector  $(X_2, X_2^{tr})$  is independent of  $X_1$ . This implies that the joint distribution function of  $(X_2, X_2^{tr}, X_1)$  factors as

$$F_{X_2, X_2^{tr}, X_1}(u_1, u_2, u_3) = F_{X_2, X_2^{tr}}(u_1, u_2)F_{X_1}(u_3).$$

Letting  $u_2 \rightarrow \infty$  in this equality, and using Kolmogorov's consistency properties for distribution functions, this implies that

$$F_{X_2, X_1}(u_1, u_3) = F_{X_2}(u_1)F_{X_1}(u_3),$$

for all  $u_1, u_3$ , which shows that  $X_2, X_1$  are independent processes, and part (a) is proved.

(2) : Assume the statements (a), (b) hold for  $i - 1$ , so that

- (a)  $X_1, \dots, X_{i-1}$  are independent processes,  
(b)  $X_{i-1}^{tr}$  is independent of the vector  $X_1, \dots, X_{i-1}$ .

Again we have that the pair of processes  $X_i^{tr}, X_i$  is obtained by thinning of the process  $X_{i-1}^{tr}$ , which with the use of (b) for  $i-1$ , shows that vector  $(X_i^{tr}, X_i)$  is independent of the vector  $(X_1, \dots, X_{i-1})$ , so that

$$F_{X_i^{tr}, X_i, X_1, \dots, X_{i-1}} = F_{X_i^{tr}, X_i} F_{X_1, \dots, X_{i-1}}.$$

Again, since the two processes  $X_i^{tr}, X_i$  are obtained by thinning of the Poisson process  $X_{i-1}^{tr}$ , they are independent so that the first factor on the right hand side factors into the univariate d.f.'s for  $X_i^{tr}, X_i$ . By (a) for  $i-1$  also the second factor of the right hand side factors into the univariate d.f.'s, so that we get

$$\begin{aligned} & F_{X_i^{tr}, X_i, X_1, \dots, X_{i-1}}(u_1, u_2, u_3, \dots, v) \\ &= F_{X_i^{tr}}(u_1) F_{X_i}(u_2) F_{X_1}(u_3), \dots, F_{X_{i-1}}(v), \end{aligned}$$

for all  $u_1, u_2, u_3, \dots, v$ . This shows that (b) holds for  $i$ .

Letting the first independent variable  $u_1 \rightarrow \infty$  in this equality and using the consistency conditions we get that

$$F_{X_i, X_1, \dots, X_{i-1}} = F_{X_i} F_{X_1}, \dots, F_{X_{i-1}}.$$

which shows that (a) holds for  $i$ .

The general result for  $\{X_i(t)\}_{i \geq 1}$  now follows by induction.  $\square$

*Proof Lemma 2.* For simplicity we skip the lower subscript  $n$  but we assume that  $a, b, c, d$  are as defined in (7).

We study the monotonicity and convexity/concavity of  $\tilde{f}$  on  $[0, \infty)$  by studying the signs of  $\tilde{f}'$  and  $\tilde{f}''$  on  $[0, \infty)$ . For  $k \geq 2$  we have

$$\tilde{f}' = a(k+1)y^k - bky^{k-1} + c,$$

$$\tilde{f}'' = y^{k-2}k(a(k+1)y - b(k-1)).$$

(i) : *The second derivative.*

Clearly  $\tilde{f}''(0) = 0$ . Factoring out  $ky^{k-2} \geq 0$ , we see that to study positive zeros and signs of  $\tilde{f}''$  is equivalent to studying the zeros and signs of

$$g(y) = a(k+1)y - b(k-1),$$

Clearly  $g(0) = -b(k-1) < 0$ ,  $g(\infty) > 0$  and  $g(y)$  has a unique root

$$y_{i.p.} = \frac{b(k-1)}{a(k+1)}.$$

From the expressions in (7) we can see that both  $a$  and  $b$  are positive and  $b > a$ , which means that  $y_{i.p.} \in (0, \infty)$ .

Thus the function  $\tilde{f}''$  is negative to the left of  $y_{i.p.}$  and positive to the right of  $y_{i.p.}$  which implies

- a)  $\tilde{f}$  is concave on  $(0, y_{i.p.})$ , convex on  $(y_{i.p.}, \infty)$ , and thus  $y_{i.p.}$  is an inflection point for  $\tilde{f}$ .

(ii) : *The first derivative.* We see that  $\tilde{f}'(0) = c > 0$ . Furthermore using the expressions for  $a, b, c$  we see that  $\tilde{f}'(1) = a(k+1) - kb + c = 0$ . From the sign change of  $\tilde{f}''$  at  $y_{i.p.}$  we have that  $\tilde{f}'$  is decreasing on  $(0, y_{i.p.})$  and increasing on  $(y_{i.p.}, \infty)$ . Now there are two possible cases:

**Case A** :  $y_{i.p.} < 1$ . In this case, the sign change of  $\tilde{f}''$  together with  $\tilde{f}'(0) = c > 0$ ,  $\tilde{f}'(1) = 0$  and the continuity of  $\tilde{f}$ , implies that for some  $y_1 < y_{i.p.}$ ,

- b')  $\tilde{f}'$  is positive on  $(0, y_1)$ , negative on  $(y_1, 1)$ , positive on  $(1, \infty)$ ,

which of course implies

- c')  $\tilde{f}$  is increasing on  $(0, y_1)$ , decreasing on  $(y_1, 1)$ , increasing on  $(1, \infty)$ .

**Case B** :  $y_{i.p.} \geq 1$ . In this case we know that  $\tilde{f}'$  is decreasing and positive on  $(0, 1)$ , decreasing and negative on  $(1, y_{i.p.})$  and increasing on  $(y_{i.p.}, \infty)$ . This implies that there is an  $y_2$  such that  $\tilde{f}'$  is negative on  $(y_{i.p.}, y_2)$  and positive on  $(y_2, \infty)$ . Thus the full statement becomes

- b'')  $\tilde{f}'$  is decreasing and positive on  $(0, 1)$ , decreasing and negative on  $(1, y_{i.p.})$ , increasing and negative on  $(y_{i.p.}, y_2)$ , increasing and positive on  $(y_2, \infty)$ .

which implies that

- c'')  $\tilde{f}$  is concave and increasing on  $(0, 1)$ , concave and decreasing on  $(1, y_{i.p.})$ , convex and decreasing on  $(y_{i.p.}, y_2)$ , convex and increasing on  $(y_2, \infty)$ .

(iii) : *The function.* We first note that  $\tilde{f}(0) = -d < 0$ , and that the expression for the coefficients  $a, b, c, d$  imply  $\tilde{f}(1) = a - b + c - d = 0$ . Now we treat the two cases separately:

**Case A:** From the sign changes of  $\tilde{f}''$  and  $\tilde{f}'$ , it follows that  $\tilde{f}$  is concave and increasing on  $(0, y_1)$ , concave and decreasing on  $(y_1, y_{i.p.})$ , convex and decreasing on  $(y_{i.p.}, 1)$ . This together with  $\tilde{f}(0) = -d < 0$ ,  $\tilde{f}(1) = 0$  implies (and in fact only the information that  $\tilde{f}$  is first increasing, then decreasing is enough) that there is a zero  $\tilde{y} \in (0, 1)$  for  $\tilde{f}$ .

**Case B:** In this case we have that  $\tilde{f}$  is increasing and concave on  $(0, 1)$ , which together with  $\tilde{f}(0) = -d < 0$ ,  $\tilde{f}(1) = 0$  implies that there is no zero for  $\tilde{f}$  in the open  $(0, 1)$ .

Finally noting that a zero  $\tilde{y}$  of  $\tilde{f}$  in  $(0, \infty)$ , corresponds, via  $\tilde{y} = 1 - \tilde{p}$ , to a zero  $\tilde{p}$  of  $f$  in  $(-\infty, 1)$ , the Lemma follows.  $\square$

*Proof of Lemma 3.* From Lemma 2, we see that

$$A_n = \left\{ \frac{b_n(k-1)}{a_n(k+1)} < 1 \right\}.$$

We will prove that

$$\frac{b_n(k-1)}{a_n(k+1)} \xrightarrow{a.s.} c < 1,$$

as  $n \rightarrow \infty$ , for some constant  $c$  (not depending on  $n$ ), from which it follows that  $P(A_n) \rightarrow 1$  as  $n \rightarrow \infty$ . Since the events  $A_n$  are independent, the second Borel-Cantelli lemma implies that  $A_n$  occurs infinitely often, almost surely.

Note that  $\{s_j\}_{j=1}^n$  and  $\{z_j\}_{j=1}^n$  in (7) are two sequences of i.i.d. random variables. Thus from the strong law of large numbers and (4)

$$\bar{s}_n \xrightarrow{a.s.} \mathbb{E}_{\theta_0} \left[ \sum_{i=1}^k x_{i1} \right] = m^{(k)}, \quad (25)$$

$$\bar{z}_n \xrightarrow{a.s.} \mathbb{E}_{\theta_0} \left[ \sum_{i=1}^k (i-1)x_{i1} \right] = \tilde{m}^{(k)}, \quad (26)$$

as  $n \rightarrow \infty$ , where

$$\begin{aligned} m^{(k)} &= \sum_{i=1}^k m_i = \sum_{i=1}^k p_0(1-p_0)^{i-1} \lambda_0 t = \lambda_0 t (1 - (1-p_0)^k), \\ \tilde{m}^{(k)} &= \sum_{i=1}^k (i-1)m_i \\ &= \lambda_0 t \frac{(k-1)(1-p_0)^{k+1} - k(1-p_0)^k + (1-p_0)}{p_0}. \end{aligned}$$

Therefore, from (7) it follows that

$$\frac{b_n(k-1)}{a_n(k+1)} \xrightarrow{a.s.} \frac{k-1}{k+1} \frac{k - (k+1)(1-p) + (1-p)^{k-1}}{(k-1) - k(1-p) + (1-p)^k} =: c,$$

as  $n \rightarrow \infty$ . One can easily prove that  $c < 1$  by considering the polynomial

$$(k-1)(1-p)^{k+1} - (k+1)(1-p)^k + (k+1)(1-p) - (k-1),$$

which is negative for all  $k > 1$  and  $0 < p < 1$ . This proves the Lemma.  $\square$

*Proof of Theorem 1 .* The proof is based on Theorem 5.1 in Lehmann and Casella (1998), and it's extension Theorem 7.1, which is based on Ralph (1962).

We have  $n$  series of  $k$  measurements  $\mathbf{x}$  of  $k \times n$  independent Poisson distributed random variables  $X_{ij}$ .

The joint log likelihood function is given by

$$\begin{aligned} l_n(\boldsymbol{\theta}|\mathbf{x}) &= \sum_{i=1}^k \sum_{j=1}^n \log f(x_{ij}|\boldsymbol{\theta}) \\ &= \sum_{j=1}^n \left\{ - \sum_{i=1}^k m_i + \sum_{i=1}^k x_{ij} \log m_i - \sum_{i=1}^k \log x_{ij}! \right\} \end{aligned}$$

We now state a sequence of results, that are necessary for the application of Theorem 5.1, and that we prove are satisfied in our case, after their statement.

- (i) For each probability mass function  $f(x_{ij}|\boldsymbol{\theta})$  we have
  - a)  $f(x_{ij}|\boldsymbol{\theta})$  are distinct.
  - b)  $f(x_{ij}|\boldsymbol{\theta})$  have common support.
- (ii) The variables  $X_{i1}, \dots, X_{in}$  are i.i.d. for each fixed  $i = 1, \dots, k$ .
- (iii) For all  $i = 1, \dots, k$  there exists an open subset  $\omega$  of  $\Omega$  containing the true parameter point  $\boldsymbol{\theta}_0$  such that for all  $X$  the probability mass function  $f(x_{ij}|\boldsymbol{\theta})$  admits all third derivatives  $(\partial^3/\partial\theta_i\partial\theta_j\partial\theta_l)f(x_{ij}|\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \omega$ .
- (iv) The first and second logarithmic derivatives of  $f(x_{ij}|\boldsymbol{\theta})$  satisfy the equations

$$\mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \log f(X_{ij}|\boldsymbol{\theta})}{\partial \theta_{\alpha}} \right] = 0, \quad (27)$$

and

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\theta}} \left[ \frac{\partial \log f(X_{ij}|\boldsymbol{\theta})}{\partial \theta_{\alpha}} \cdot \frac{\partial \log f(X_{ij}|\boldsymbol{\theta})}{\partial \theta_{\beta}} \right] \\ = \mathbb{E}_{\boldsymbol{\theta}} \left[ -\frac{\partial^2}{\partial \theta_{\alpha} \partial \theta_{\beta}} \log f(X_{ij}|\boldsymbol{\theta}) \right]. \end{aligned} \quad (28)$$

(v) The elements of information matrix  $[\mathbf{I}(\boldsymbol{\theta})]_{\alpha\beta}$  are finite, and the information matrix is positive definite for all  $\boldsymbol{\theta} \in \omega$ .

(vi) There exist functions  $M_{ijl}$  such that

$$\left| \frac{\partial^3}{\partial \theta_{\alpha} \partial \theta_{\beta} \partial \theta_{\gamma}} \log f(X_{ij}|\boldsymbol{\theta}) \right| \leq M_{ijl}(X_{ij}),$$

for all  $\boldsymbol{\theta} \in \omega \subset \Omega$  and  $\mathbb{E}_{\boldsymbol{\theta}_0} [M_{\alpha\beta\gamma}(X_{ij})] < \infty$ .

*Proof of (i) – (ii):* The statements (i) and (ii)a) follow from the expression (1), and (ii)b) follows from the statement of this Lemma.

*Proof of (iii):* Let  $\omega$  be, for example, the open subset  $\omega \subset \Omega$  such that  $g\{\theta = (p, \lambda) \in \omega : p_0 - a < p < p_0 + a; \lambda_0 - b < \lambda < \lambda_0 + b\}$ , where  $a = \min(0.1 \cdot p_0; 0.1 \cdot (1 - p_0))$  and  $b = 0.1 \cdot \lambda$ . For such  $\omega$  (iv) follows.

*Proof of (iv):*  $\log f(X_{ij}|\boldsymbol{\theta})$  and its first and second derivatives are given by:

$$\log f(X_{ij}|\boldsymbol{\theta}) = -m_i + X_{ij} \log m_i - \log X_{ij}!, \quad (29)$$

$$\frac{\partial \log f(X_{ij}|\boldsymbol{\theta})}{\partial \lambda} = \frac{X_{ij} - p(1-p)^{i-1}\lambda t}{\lambda}, \quad (30)$$

$$\frac{\partial^2 \log f(X_{ij}|\boldsymbol{\theta})}{\partial \lambda^2} = -\frac{X_{ij}}{\lambda^2}, \quad (31)$$

$$\begin{aligned} \frac{\partial \log f(X_{ij}|\boldsymbol{\theta})}{\partial p} &= -((1-p)^{i-1} - p(i-1)(1-p)^{i-2})\lambda t \\ &\quad + \frac{X_{ij}}{p} - \frac{X_{ij}(i-1)}{(1-p)}, \end{aligned} \quad (32)$$

$$\begin{aligned} \frac{\partial^2 \log f(X_{ij}|\boldsymbol{\theta})}{\partial p^2} &= \lambda t(2(i-1)(1-p)^{i-2} - p(i-1)(i-2)(1-p)^{i-3}) \\ &\quad - \frac{X_{ij}}{p^2} - \frac{X_{ij}(i-1)}{(1-p)^2}, \end{aligned} \quad (33)$$

$$\begin{aligned} \frac{\partial^2 \log f(X_{ij}|\boldsymbol{\theta})}{\partial p \partial \lambda} &= \frac{\partial^2 \log f(X_{ij}|\boldsymbol{\theta})}{\partial \lambda \partial p} = \\ &= -((1-p)^{i-1} - p(i-1)(1-p)^{i-2})t. \end{aligned} \quad (34)$$



Substituting these expressions into (27) and (28) and using  $\mathbb{E}_{\boldsymbol{\theta}} [X_{ij}] = p(1 - p)^{i-1}\lambda t$ , we prove the equalities.

*Proof of (v):* From (iv) we have

$$[\mathbf{I}_{(i)}(\boldsymbol{\theta})]_{\alpha\beta} = \mathbb{E}_{\boldsymbol{\theta}} \left[ -\frac{\partial^2}{\partial\theta_{\alpha}\partial\theta_{\beta}} \log f(X_{ij}|\boldsymbol{\theta}) \right]$$

so that, for the elements of  $[\mathbf{I}_{(i)}(\boldsymbol{\theta})]$  we have:

$$\begin{aligned} [\mathbf{I}_{(i)}(\boldsymbol{\theta})]_{pp} &= \lambda t y^{i-4} (1-y)^2 (-y^4 + 2y^3 + (i^2 - 1)y^2 + 2(i - i^2)y) \\ &\quad + \lambda t y^{i-4} (1-y)^2 (1-i)^2, \\ [\mathbf{I}_{(i)}(\boldsymbol{\theta})]_{\lambda p} &= t(y^{i-1} + (i-1)y^{i-1} - (i-1)y^{i-2}), \\ [\mathbf{I}_{(i)}(\boldsymbol{\theta})]_{\lambda\lambda} &= t \frac{y^{i-1} - y^i}{\lambda}, \end{aligned} \tag{35}$$

with  $y = 1 - p$ .

Further, using Sylvester's criterion, we can show that: (a)  $[\mathbf{I}(\theta)]_{\lambda\lambda} > 0$  and (b)  $[\mathbf{I}(\theta)]_{pp} [\mathbf{I}(\theta)]_{\lambda\lambda} - [\mathbf{I}(\theta)]_{\lambda p}^2 > 0$  for all  $\boldsymbol{\theta}$  in  $\Omega$ .

(a) is obvious, since  $y = 1 - p \in (0, 1)$ .

For (b), after simplification the inequality  $[\mathbf{I}(\theta)]_{pp} [\mathbf{I}(\theta)]_{\lambda\lambda} - [\mathbf{I}(\theta)]_{\lambda p}^2 > 0$  can be written as

$$t^2 y^{i-4} (1-y)^2 (-y^4 + 2y^3 + (i^2 - 1)y^2 + 2(i - i^2)y + (1 - i)^2) > 0,$$

and we get the following inequality to prove:

$$-y^4 + 2y^3 + (i^2 - 1)y^2 + 2(i - i^2)y + (1 - i)^2 > 0.$$

The proof is quite straightforward and can be done by studying the monotonicity and convexity/concavity of the polynomial on  $(0, 1)$  by analysing the signs of its first three derivatives on  $(0, 1)$ .

*Proof of (vi):* From (30) – (34) it follows, that third partial derivatives of  $\log f(X_{ij}|\boldsymbol{\theta})$  exist and finite in  $\omega$ . In our case  $M_{ijl}$ , for example, is

$$M_{ijl}(X_{ij}) = \sup_{\boldsymbol{\theta} \in \omega} \left| \frac{\partial^3}{\partial\theta_i \partial\theta_j \partial\theta_l} \log f(X_{ij}|\boldsymbol{\theta}) \right|.$$

This completes the proof. □

*Proof (Lemma 4)*

For  $\sigma_p^2(p, \lambda)$  we have

$$\begin{aligned}\sigma_p^2(p, \lambda) &= \frac{(1 - (1 - p)^k)(1 - p)p^2}{\lambda t q(p)} \\ &= s(k, p) \frac{(1 - p)p^2}{\lambda t},\end{aligned}\tag{36}$$

with  $s(k, p) = \frac{(1 - (1 - p)^k)}{q(p)}$ . Let us consider the function

$$s(x) = \frac{(1 - y^x)}{\tilde{q}(y)},\tag{37}$$

with support in  $(0, \infty)$ ,  $y = 1 - p$  a parameter in  $(0, 1)$ , and  $\tilde{q}(y) = q(p)$ , (cf. (13)). The first derivative of  $s(x)$  is

$$s'(x) = \frac{y^x}{\tilde{q}(y)^2} s_1(x),\tag{38}$$

where  $s_1(x) = \log(y)(1 - y^x)^2 + \frac{(y-1)^2}{y} x(\log(y)x + 2(1 - y^x))$ .

Since  $\frac{y^x}{\tilde{q}(y)^2} > 0$ , we have to prove that  $s_1(x) < 0$  on  $x > 1$  for  $y \in (0, 1)$ . The first derivative of  $s_1(x)$  is

$$s'_1(x) = 2(1 - y^x)s_2(x),\tag{39}$$

where  $s_2(x) = -y^x \log^2(y) + \frac{(y-1)^2}{y} (x \log(y) + 1)$ .

Further,  $s'_2(x)$  is

$$s'_2(x) = \log(y)s_3(x),\tag{40}$$

where  $s_3(x) = -y^x \log^2(y) + \frac{(y-1)^2}{y}$ .

The function  $s_3(x)$  is an increasing function of  $x$  for  $x \geq 1$  for  $y \in (0, 1)$ . Let us fix  $x = 1$  and now consider  $s_3$  as a function on  $y$  with support  $y \in (0, 1)$  and analyze its behaviour. We have:

$$s_3(y) = -y \log^2(y) + \frac{(y-1)^2}{y},\tag{41}$$

and its first 3 derivatives are:

$$s_3^{(1)}(y) = -\log^2(y) - 2\log(y) + 1 - \frac{1}{y^2},\tag{42}$$

$$s_3^{(2)}(y) = -2\frac{\log(y)}{y} - \frac{2}{y} + \frac{2}{y^3},\tag{43}$$

$$s_3^{(3)}(y) = 2\frac{\log(y)}{y^2} - \frac{6}{y^4},\tag{44}$$

From (44) we see that  $s_3^{(3)}(y) < 0$ . It follows that  $s_3^{(2)}(y)$  is monotonically decreasing function on  $(0, 1)$ . In addition, from (43),  $s_3^{(2)}(y) \rightarrow +\infty$  if  $y \rightarrow 0$  and  $s_3^{(2)}(1) = 0$ . So that,  $s_3^{(2)}(y) > 0$  on  $(0, 1)$  and  $s_2^{(1)}(y)$  is monotonically increasing function on  $(0, 1)$ . Taking into account from (42) that  $s_3^{(1)}(y) \rightarrow -\infty$  if  $y \rightarrow 0$  and  $s_3^{(1)}(1) = 0$ , we get that  $s_3^{(1)}(y) < 0$  on  $(0, 1)$ . It gives us, that  $s_3(y)$  is monotonically decreasing and since from (41)  $s_3(y) \rightarrow +\infty$  if  $y \rightarrow 0$  and  $s_3^{(1)}(1) = 0$ ,  $s_3(y)$  is positive on  $(0, 1)$ .

Let's go back to  $s_3(x)$  in (40). As it was mentioned above,  $s_3(x)$  is increasing function of  $x$  on  $[1, +\infty)$ . Since  $s_3(1) > 0$  for every  $y \in (0, 1)$ ,  $s_3(x)$  is positive for  $x \geq 1$ . Using (40), it follows, that  $s_2'(x) < 0$  and  $s_2(x)$  is decreasing on  $x \geq 1$ . Again, considering function  $s_2(x)$  as the function of  $y$  with support  $(0, 1)$  for  $x = 1$ , in exactly the same way as it was done for  $s_3(x)$ , we can prove that  $s_2(y) < 0$  on  $(0, 1)$ . Taking into account, that  $s_2(x)$  is decreasing function of  $x$  on  $[1, +\infty)$  and using (39) we get  $s_1'(x) < 0$  on  $[1, +\infty)$ . So that  $s_1(x)$  is decreasing on  $[1, +\infty)$ .

It remains to prove that  $s_1(y) < 0$  for  $y \in (0, 1)$ . Again, since  $s_1(x)$  is decreasing on  $[1, +\infty)$ , it is enough to show that  $s_1(y) < 0$  on  $y \in (0, 1)$  for  $x = 1$ . It can be done in exactly the same way as for  $s_3(y)$  and  $s_2(y)$ .

Thus we have proved that  $s(x)$  is decreasing function of  $x$  on  $[q, +\infty)$ . From the expression (36) we conclude, that the standard deviation  $\sigma_{\hat{p}_n}$  is a decreasing function of  $k$  for  $k = 2, 3, \dots, +\infty$ .  $\square$

*Proof of Corollary 1.* Expanding (20) in a Taylor series around the point  $(p_0, \chi_0)$  of the true values of  $p$  and  $\chi$ , we get:

$$\begin{aligned} \mu(p, \chi) &= \mu(p_0, \chi_0) + \frac{\partial \mu}{\partial p}(p_0, \chi_0)(p - p_0) + \frac{\partial \mu}{\partial \chi}(p_0, \chi_0)(\chi - \chi_0) \\ &+ R_2(p, \chi), \end{aligned} \quad (45)$$

where  $R_2(p, \chi)$  is the Lagrange Remainder:

$$\begin{aligned} R_2(p, \chi) &= \frac{\partial^2 \mu}{2\partial p^2}(\tilde{p}, \tilde{\chi})(p - p_0)^2 + \frac{\partial^2 \mu}{\partial p \partial \chi}(\tilde{p}, \tilde{\chi})(p - p_0)(\chi - \chi_0) \\ &+ \frac{\partial^2 \mu}{2\partial \chi^2}(\tilde{p}, \tilde{\chi})(\chi - \chi_0)^2, \end{aligned}$$

where  $\tilde{p}$  is the number between  $p_0$  and  $p$  and  $\tilde{\chi}$  is between  $\chi_0$  and  $\chi$ .

Assume that there has been made  $n$  measurements for  $(\hat{p}_n, \hat{\lambda}_n)$  and  $n'$  measurements for  $\hat{\chi}_{n'}$ , and that  $(\hat{p}_n, \hat{\lambda}_n)$  and  $\hat{\chi}_{n'}$  are statistically independent. Let  $n' = \lceil \gamma n \rceil$ , with  $\gamma$  a proportionality factor that we introduce for convenience.

Multiplying (45) by  $\sqrt{n}$  and letting  $p = \hat{p}_n$  and  $\chi = \hat{\chi}_{n'}$  we get

$$\begin{aligned}\sqrt{n}(\mu(\hat{p}_n, \hat{\chi}_{n'}) - \mu(p_0, \chi_0)) &= \frac{\partial \mu}{\partial p}(p_0, \chi_0) \sqrt{n}(\hat{p}_n - p_0) \\ &+ \frac{\partial \mu}{\partial \chi}(p_0, \chi_0) \sqrt{n}(\hat{\chi}_{n'} - \chi_0) \\ &+ \sqrt{n}R_2(\hat{p}_n, \hat{\chi}_{n'}).\end{aligned}\quad (46)$$

From the asymptotic normality of the estimators  $\hat{p}_n$  and  $\hat{\chi}_{n'}$  for the first two terms we have

$$\sqrt{n}(\hat{p}_n - p_0) \xrightarrow{d} \mathcal{N}g(0, \sigma_p^2 g), \quad (47)$$

as  $n \rightarrow \infty$  and

$$\begin{aligned}\sqrt{n}(\hat{\chi}_{n'} - \chi_0) &= \sqrt{\frac{n}{n'}} \sqrt{n'}(\hat{\chi}_{n'} - \chi_0) \\ &= \sqrt{\frac{n}{\lceil \gamma n \rceil}} \sqrt{n'}(\hat{\chi}_{n'} - \chi_0) \xrightarrow{d} \mathcal{N}g(0, \frac{\sigma_{\tilde{\chi}}^2}{\gamma} g),\end{aligned}\quad (48)$$

as  $n \rightarrow \infty$ , since  $\lim_{n \rightarrow \infty} \frac{n}{\lceil \gamma n \rceil} = \frac{1}{\gamma}$ .

Let us show first that

$$\sqrt{n}R_2(\hat{p}_n, \hat{\chi}_{n'}) \xrightarrow{p} 0, \quad (49)$$

as  $n \rightarrow \infty$ .

We can rewrite the remainder term  $\sqrt{n}R_2(\hat{p}_n, \hat{\chi}_{n'})$  as

$$\begin{aligned}\sqrt{n}R_2(\hat{p}_n, \hat{\chi}_{n'}) &= \\ &= g\left(\frac{1}{2} \frac{\partial^2 \mu}{\partial p^2}(\tilde{p}_n, \tilde{\chi}_{n'})\right)(\hat{p}_n - p_0) \\ &+ \frac{1}{2} \frac{\partial^2 \mu}{\partial p \partial \chi}(\tilde{p}_n, \tilde{\chi}_{n'}) (\hat{\chi}_{n'} - \chi_0) g \sqrt{n}(\hat{p}_n - p_0), \\ &+ g\left(\frac{1}{2} \frac{\partial^2 \mu}{\partial \chi^2}(\tilde{p}_n, \tilde{\chi}_{n'})\right)(\hat{\chi}_{n'} - \chi_0) \\ &+ \frac{1}{2} \frac{\partial^2 \mu}{\partial p \partial \chi}(\tilde{p}_n, \tilde{\chi}_{n'}) (\hat{p}_n - p_0) g \sqrt{n}(\hat{\chi}_{n'} - \chi_0),\end{aligned}$$

where  $\tilde{p}_n$  is between  $p_0$  and  $\hat{p}_n$  and  $\tilde{\chi}_{n'}$  is between  $\chi_0$  and  $\hat{\chi}_{n'}$  and it can be shown that

$$\frac{1}{2} \frac{\partial^2 \mu}{\partial p^2}(\tilde{p}_n, \tilde{\chi}_{n'}) (\hat{p}_n - p_0) + \frac{1}{2} \frac{\partial^2 \mu}{\partial p \partial \chi}(\tilde{p}_n, \tilde{\chi}_{n'}) (\hat{\chi}_{n'} - \chi_0) \xrightarrow{p} 0, \quad (50)$$

and

$$\frac{1}{2} \frac{\partial^2 \mu}{\partial \chi^2}(\tilde{p}_n, \tilde{\chi}_{n'}) (\hat{\chi}_{n'} - \chi_0) + \frac{1}{2} \frac{\partial^2 \mu}{\partial p \partial \chi}(\tilde{p}_n, \tilde{\chi}_{n'}) (\hat{p}_n - p_0) \xrightarrow{p} 0 \quad (51)$$

as  $n \rightarrow \infty$ .

The first factor  $\frac{1}{2} \frac{\partial^2 \mu}{\partial p^2}(\tilde{p}_n, \tilde{\chi}_{n'})$  in the first term in (50) converges to  $\frac{1}{2} \frac{\partial^2 \mu}{\partial p^2}(p_0, \chi_0)$  in probability by the consistency of estimators  $\hat{p}_n$  and  $\hat{\chi}_{n'}$ , since  $\tilde{p}_n \in (p_0, \hat{p}_n)$  and  $\tilde{\chi}_{n'} \in (\chi_0, \hat{\chi}_{n'})$  and via the continuous mapping theorem, since  $\frac{\partial^2 \mu}{2 \partial p^2}$  is a continuous function. The second factor  $(\hat{p}_n - p_0)$  converges in probability to zero by the consistency of  $\hat{p}_n$ . Using Slutsky's theorem, this implies that the first term in (50) converges to zero in distribution and therefore in probability. The same holds for the second term. Since both terms converge to zero in probability, using Slutsky's theorem again, we get that their sum goes to zero in distribution and also in probability. We can prove (51) in the similar way. Using Slutsky's theorem and (47) and (48) we prove (49).

Next, using independence of estimators  $\hat{p}_n$  and  $\hat{\chi}_{n'}$  and their asymptotic properties (47) and (48) we get the following asymptotic result for  $\hat{\mu}$ :

$$\sqrt{n}(\mu(\hat{p}_n, \hat{\chi}_{n'}) - \mu(p_0, \chi_0)) \xrightarrow{d} \mathcal{N}(0, \sigma_{\hat{\mu}}^2),$$

where

$$\sigma_{\hat{\mu}}^2 = \left[ \frac{\partial \mu}{\partial p}(p_0, \chi_0) \right]^2 \sigma_{\hat{p}}^2(p_0, \lambda_0) + \left[ \frac{\partial \mu}{\partial \chi}(p_0, \chi_0) \right]^2 \frac{\sigma_{\hat{\chi}}^2}{\gamma}$$

□

*Proof of Corollary 2.* First, let us first show that

$$S_n^2 \xrightarrow{p} \sigma_{\hat{\mu}}^2, \quad (52)$$

as  $n \rightarrow \infty$ .

The first factor  $\left[ \frac{\partial \mu}{\partial p}(\hat{p}_n, \hat{\chi}_{n'}) \right]^2$  in the first term in (22) converges in probability to  $\left[ \frac{\partial \mu}{\partial p}(p_0, \chi_0) \right]^2$  by the consistency of estimators  $\hat{p}_n$  and  $\hat{\chi}_{n'}$  and via continuous mapping theorem, since  $\left[ \frac{\partial \mu}{\partial p}(\hat{p}_n, \hat{\chi}_{n'}) \right]^2$  is continuous function. The same holds for  $\sigma_{\hat{p}}^2(\hat{p}_n, \hat{\lambda}_n)$ , which converges in probability to  $\sigma_{\hat{p}}^2(p_0, \lambda_0)$ . Using Slutsky's theorem, this implies that the first term in (22) converges in distribution to  $\left[ \frac{\partial \mu}{\partial p}(p_0, \chi_0) \right]^2 \sigma_{\hat{p}}^2(p_0, \lambda_0)$  and therefore in probability.

The first factor  $\frac{1}{\gamma} \left[ \frac{\partial \mu}{\partial \chi}(\hat{p}_n, \hat{\chi}_{n'}) \right]^2$ , in the second term converges to  $\frac{1}{\gamma} \left[ \frac{\partial \mu}{\partial \chi}(p_0, \chi_0) \right]^2$  in probability by the consistency of estimators  $\hat{p}_n$  and  $\hat{\chi}_{n'}$  and via continuous mapping theorem. Second factor  $\hat{\sigma}_{\hat{\chi}}^2$  of the second term is the pooled variance based on three series of the same numbers of i.i.d. measurements. Therefore  $\hat{\sigma}_{\hat{\chi}}^2$  is a consistent estimator of the variance of  $\hat{\chi}$ , which means that  $\hat{\sigma}_{\hat{\chi}}^2$  converges in probability to  $\sigma_{\hat{\chi}}^2$ . By Slutsky's theorem, second term converges in distribution to  $\frac{1}{\gamma} \left[ \frac{\partial \mu}{\partial \chi}(p_0, \chi_0) \right]^2 \sigma_{\hat{\chi}}^2$  and therefore in probability.

It has been shown that both terms in (22) converge in probability. Using Slutsky's theorem again we prove the statement (52).

Second, from Corollary 1,

$$\sqrt{n}(\mu(\hat{p}_n, \hat{\chi}_{n'}) - \mu(p_0, \chi_0)) \xrightarrow{d} \mathcal{N}(0, \sigma_{\hat{\mu}}^2),$$

dividing the left hand side by  $S_n$ , using the fact that  $S_n \xrightarrow{p} \sigma_{\hat{\mu}}$ , as  $n \rightarrow \infty$  and via Slutsky's theorem, we prove (23).

□

## References

- [1] Anevski D. & Hall-Wilton R. (2012). Statistical methods for energy determination in neutron detector systems. Technical report, Mathematical Sciences Lund University and ESS, 2012.
- [2] Assuncao R. M. & Ferrari P. A. (2007). Independence of thinned processes characterizes the Poisson process: An elementary proof and a statistical application. *TEST* **16**, 333–345.
- [3] Gut A. (2009). *An Intermediate Course in Probability*, Springer Dordrecht Heidelberg London New York.
- [4] Kulkarni V. G. (2009). *Modeling and Analysis of Stochastic Systems*, CRC Press.
- [5] Lehmann P. A. & Casella P. A. (1998). *Theory of Point Estimation*, Springer-Verlag, New York.
- [6] Schmitt H. W., Block R. C. & Bailey R, L. (1959). Total Neutron cross section of  $B^{10}$  in the thermal neutron energy range. *Nuclear Physics* **17**, 109–115.
- [7] Ralph A., Bradley A. & Cart J. J. (1962). The asymptotic properties of ML estimators when sampling from associated populations. *Biometrika* **49**, 205–214.
- [8] van der Vaart A.W. (1998). *Asymptotic Statistics*, Cambridge Univ. Press, New York.

